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APPROXIMATE THEORY FOR LOCALLY LOADED PLANE ORTHOTROPIC BEAMS†

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Abstract—Governing equations for the orthotropic, rectangular strip are investigated to obtain solutions illustrating the relationship between classical beam theory and higher order theories. The exact equations of generalized plane elasticity are reduced to coupled sets of ordinary differential equations by using series representations in Legendre polynomials for all dependent variables. The coupled differential equations are obtained in a form such that a proper truncation scheme to extract any higher order theory is easily formulated. Sample problems involving uniformly and locally loaded beams are worked out using typical values for the material constants. Comparison of results with existing solutions shows that problems involving slowly varying loads with material orthotropy can be handled by classical thin beam theory; however, local loadings, coupled with materials orthotropy, preclude the use of any classical beam theory, even for thin beams.

NOTATION

l	length of rectangular strip
h	thickness of strip
x, y	axial, thickness coordinates
η	dimensionless thickness coordinate
N, M, Q	stress resultants of classical beam theory
$S^{(n)}$	higher order stress coefficients
U_A, W_A, β	deformation variables of classical beam theory
$U^{(n)}$	higher order deformation coefficients
$P_n(\eta)$	Legendre polynomial of integral order n
$Q_n(\eta)$	$= P_n(\eta) - P_{n-2}(\eta)$
$E_x, E_y, \nu_{xy}, \nu_{yx}, G_{xy}$	orthotropic material properties
p, p^*, q, q^*	specified lateral surface tractions
a	local loading parameter

INTRODUCTION

IT IS well known that classical beam theory is not valid for beams with large thickness-to-length ratios; also the effects of localized loading are not accurately predicted by classical beam theory even for small thickness-to-length ratios. These effects may be greatly magnified by material anisotropy.

In this work, a higher order beam theory is established by suitably truncating Legendre polynomial series representations of the 2-D plane elasticity solution. Many authors have used just such Legendre polynomial series representations to construct approximate solutions to various elasticity problems; for example, we note the work of Mindlin and

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Medick [1], Medick [2] and Hertelendy [3] concerned with higher order dynamic effects in plates and rods. The inclusion of higher order terms in the truncation yields a beam theory containing the main corrections to classical beam theory due to thickness effects, local loading effects and material anisotropy. It is recognized by the authors that with the advent of modern high speed computers, efficient numerical techniques have been developed to solve such plane problems; however, it is felt that there still exists some justification for studying approximate analytical solutions to gain insight into effects of varying values of material properties and to ascertain regions of rapidly varying stress and deformation.

Previous investigators of anisotropy effects in plane elastic strips concentrated their efforts on distributed polynomial loadings [4, 5]. In general, these previous investigators concluded that for these kinds of loadings, the effect of anisotropy shows up mainly in the value used for the Young's Modulus in classical beam theory; other than this effect, they detected no deviations in the stress and deformation fields (from the predictions of classical beam theory) that were of practical engineering significance.

A higher order elastic beam theory has been developed for isotropic materials; it has been shown that this new theory was capable of predicting deviations from classical beam theory due to localized loading and due to thickness effects [6]. In the first part of the work to follow, we extend the solution procedure developed in Ref. [6] to the case of orthotropic beams; the main purpose of this study is to examine the orthotropic beam problem in the presence of local loading. We show that material anisotropy, coupled with local loadings, results in classical beam theory severely underestimating maximum stresses even for relatively small thickness-to-length ratios. On the other hand, we reinforce a conclusion of Ref. [4] that for slowly varying loading, material orthotropy does not prevent the use of a slightly modified classical theory for thin beams.

DEVELOPMENT OF HIGHER ORDER BEAM THEORY

Using the notation of Fig. 1, the governing generalized plane stress equations including material orthotropy are: ($\eta = 2y/h$)

$$
\begin{aligned}
&\frac{\partial \sigma_x}{\partial x}+\frac{2}{h}\frac{\partial \sigma_{xy}}{\partial \eta}=0 \qquad && \frac{2E_y}{h}\frac{\partial u_y}{\partial \eta}=\sigma_y-\nu_{yx}\sigma_x \\
&\frac{\partial \sigma_{xy}}{\partial x}+\frac{2}{h}\frac{\partial \sigma_y}{\partial \eta}=0 \qquad && \frac{2}{h}\frac{\partial u_x}{\partial \eta}+\frac{\partial u_y}{\partial x}=\frac{\sigma_{xy}}{G_{xy}} \\
&E_x\frac{\partial u_x}{\partial x}=\sigma_x-\nu_{xy}\sigma_y \qquad && \frac{\nu_{yx}}{E_y}=\frac{\nu_{xy}}{E_x}.
\end{aligned}
\tag{1}
$$

Associated with these equations boundary conditions are

$$
\begin{aligned}
&\sigma_{xy} \text{ or } u_x \text{ and } \sigma_y \text{ or } u_y \text{ specified on } \eta = \pm 1, \text{ all } x \\
&\sigma_x \text{ or } u_x \text{ and } \sigma_{xy} \text{ or } u_y \text{ specified on } x = \text{const. edges, all } \eta.
\end{aligned}
\tag{2}
$$

Following [3], we assume convergent series solutions for all of the dependent variables in Legendre polynomials:

$$\begin{aligned}
\sigma_x(x,\eta) &= \sum_{n=0}^{\infty} S_x^{(n)} P_n(\eta) \\
\sigma_{xy}(x,\eta) &= \sum_{n=0}^{\infty} S_{xy}^{(n)} Q_n(\eta) = \sum_{n=0}^{\infty} (S_{xy}^{(n)} - S_{xy}^{(n+2)}) P_n(\eta) \\
\sigma_y(x,\eta) &= \sum_{n=0}^{\infty} S_y^{(n)} Q_n(\eta) = \sum_{n=0}^{\infty} (S_y^{(n)} - S_y^{(n+2)}) P_n(\eta) \\
u_x(x,\eta) &= \sum_{n=0}^{\infty} U_x^{(n)} Q_n(\eta) = \sum_{n=0}^{\infty} (U_x^{(n)} - U_x^{(n+2)}) P_n(\eta) \\
u_y(x,\eta) &= \sum_{n=0}^{\infty} U_y^{(n)} Q_n(\eta) = \sum_{n=0}^{\infty} (U_y^{(n)} - U_y^{(n+2)}) P_n(\eta).
\end{aligned} \tag{3}$$

In the above, we have defined the complete set of functions $Q_n(\eta)$ as follows:

$$Q_0(\eta) = P_0(\eta);\; Q_1(\eta) = P_1(\eta);\; Q_n(\eta) = P_n(\eta) - P_{n+2}(\eta) \qquad n > 1. \tag{4}$$

Note that since $Q_n(\pm 1) = 0$ for $n > 1$, the first two coefficients of each series representation in $Q_n(\eta)$ completely describe the function on the lateral surfaces of the beam; hence lateral surface boundary conditions are exactly satisfied by proper choice of certain coefficient functions having superscripts zero and one.

Substituting equations (3) into equations (1) and making use of certain recurrence relations involving Legendre polynomials removes the η dependence and yields the set of ordinary differential equations governing the coefficient functions as:

$$S_{xy}^{(n+1)} = -\frac{h}{2(2n+1)} \frac{\mathrm{d}S_x^{(n)}}{\mathrm{d}x} \qquad n \geq 0 \tag{5}$$

$$S_y^{(n+1)} = -\frac{h}{2(2n+1)} \frac{\mathrm{d}}{\mathrm{d}x}(S_{xy}^{(n)} - S_{xy}^{(n+2)}) \tag{6}$$

$$S_x^{(n)} = E_x \frac{\mathrm{d}}{\mathrm{d}x}(U_x^{(n)} - U_x^{(n+2)}) + \nu_{xy}(S_y^{(n)} - S_y^{(n+2)}) \tag{7}$$

$$U_y^{(n+1)} = \frac{h}{2E_y(2n+1)} \left\{ S_y^{(n)} - S_y^{(n+2)} - \nu_{yx} S_x^{(n)} \right\} \tag{8}$$

$$U_x^{(n+1)} = \frac{h}{2(2n+1)} \left\{ \frac{S_x^{(n)} - S_{xy}^{(n+2)}}{G_{xy}} - \frac{\mathrm{d}}{\mathrm{d}x}(U_y^{(n)} - U_y^{(n+2)}) \right\}. \tag{9}$$

Equations (5)–(9) represent a five-fold infinite set of coupled equations which, when solved, yield the exact solution to the plane elasticity problem. To practically obtain a solution, of course, requires that a suitable truncation procedure be established. In Ref. [6], equations for the isotropic case [similar to equations (5)–(9)] were shown to decouple to the extent that all unknown coefficient functions were determined exactly in terms of the shear stress coefficients $S_{xy}^{(n)}$; further, a set of coupled equations involving only the coefficients $S_{xy}^{(n)}$ were developed. Therefore, in Ref. [6], the truncation or closure, of the infinite set of equations was established based only on a truncation of the shear stress series. In this work we proceed in a slightly different manner; that is, we first establish a truncation procedure for the set of equations (5)–(9), and then, only in solving a particular problem do we obtain the relevant governing equations in terms of shear stress variables.

We consider an Mth order approximate theory to consist of the exact first $M+1$ contributing equations from each of equations (5)–(9); these $5M+5$ equations contain $5M+13$ coefficient functions of which four are specified by virtue of satisfying boundary conditions on $\eta = \pm 1$. As written, the Mth order theory is not complete in that there are four additional functions $U_x^{(M+2)}$, $U_y^{(M+2)}$, $S_y^{(M+2)}$, $S_{xy}^{(M+2)}$ for which we have not written governing equations. The form of equations (5)–(9) suggests the following closure scheme. The $5M+5$ equations as written are exact and illustrate all of the couplings between the variables that appear; the closure is effected by using equations (5), (6), (8) and (9) for $n = M+1$ to obtain the needed four additional equations. The approximate nature of the theory is introduced by *neglecting all terms in these four additional equations containing any functions not already introduced in the original* $5M$ *equations*. Thus, to obtain a deterministic Mth order theory, we write, in addition to the $5M+5$ exact equations, the four approximate equations

$$S_{xy}^{(M+2)} \approx 0; \qquad S_y^{(M+2)} \approx -\frac{h}{2(2M+3)}\frac{\mathrm{d}}{\mathrm{d}x}S_{xy}^{(M+1)} \tag{10}$$

$$U_y^{(M+2)} \approx \frac{h}{2E_y(2M+3)}S_y^{(M+1)}; \qquad U_x^{(M+2)} \approx \frac{h}{2(2M+3)}\left\{\frac{S_{xy}^{(M+1)}}{G_{xy}} - \frac{\mathrm{d}U_y^{(M+1)}}{\mathrm{d}x}\right\}. \tag{11}$$

Some additional comments concerning this truncation scheme are found in Ref. [7].

As noted previously, boundary conditions on the lateral surfaces are satisfied by specifying four of the eight coefficient functions $S_{xy}^{(0)}$, $S_{xy}^{(1)}$, $S_y^{(0)}$, $S_y^{(1)}$, $U_x^{(0)}$, $U_x^{(1)}$, $U_y^{(0)}$, $U_y^{(1)}$ depending upon the problem considered. Boundary conditions on the $x =$ const. end surfaces are easily satisfied up to the order of the Mth approximation by using the orthogonality of the Legendre polynomials; thus on the end faces we must specify

$$S_x^{(n)} \text{ or } U_x^{(n)} - U_x^{(n+2)} \quad \text{and} \quad S_{xy}^{(n)} - S_{xy}^{(n+2)} \text{ or } U_y^{(n)} - U_y^{(n+2)} \qquad n = 0, 1, \ldots M. \tag{12}$$

EQUATIONS FOR THIRD ORDER APPROXIMATION

In this section are presented the governing equations for a third order approximation in the sense defined above; such a theory may be sufficient to obtain analytical results for many problems of engineering interest. The following definitions from classical beam theory are noted

$$\begin{aligned} p(x) &= \sigma_{xy}(x,1)+\sigma_{xy}(x,-1); \qquad p^*(x) = \sigma_{xy}(x,1)-\sigma_{xy}(x,-1) \\ q(x) &= \sigma_y(x,1)+\sigma_y(x,-1); \qquad q^*(x) = \sigma_y(x,1)-\sigma_y(x,-1) \end{aligned} \tag{13}$$

$$\begin{aligned} N(x) &= \int_{-h/2}^{h/2} \sigma_x \,\mathrm{d}y; \qquad M(x) = \int_{-h/2}^{h/2} y\sigma_x \,\mathrm{d}y \\ Q(x) &= \int_{-h/2}^{h/2} \sigma_{xy} \,\mathrm{d}y; \qquad \beta(x) = \frac{12}{h^3}\int_{-h/2}^{h/2} yu_u \,\mathrm{d}y \\ U_A(x) &= \frac{1}{h}\int_{-h/2}^{h/2} u_x \,\mathrm{d}y; \qquad W_A(x) = \frac{1}{h}\int_{-h/2}^{h/2} u_y \,\mathrm{d}y. \end{aligned} \tag{14}$$

In terms of the coefficient functions used in this study, we have

$$p(x) = 2S_{xy}^{(0)}; \quad p^*(x) = 2S_{xy}^{(1)}; \quad q(x) = 2S_y^{(0)}; \quad q^*(x) = 2S_y^{(1)}$$

$$U_x^{(1)} - U_x^{(3)} = \frac{h}{2}\beta(x); \quad U_x^{(0)} - U_x^{(2)} = U_A(x); \quad U_y^{(0)} - U_y^{(2)} = W_A(x) \tag{15}$$

$$S_x^{(0)} = \frac{N(x)}{h}; \quad S_x^{(1)} = \frac{6M}{h^2}; \quad S_{xy}^{(0)} - S_{xy}^{(2)} = \frac{Q(x)}{h}.$$

In the study to follow, we assume that tractions are prescribed on $\eta = \pm 1$ so that $p(x)$, $p^*(x)$, $q(x)$, $q^*(x)$ are given functions. Specializing equations (5)–(11) for a third order theory ($M = 3$) yields, after some algebraic manipulations, the following governing equations:

Extension

$$\frac{\mathrm{d}N(x)}{\mathrm{d}x} + p^*(x) = 0$$

$$E_x h \frac{\mathrm{d}}{\mathrm{d}x}(U_x^{(0)} - U_x^{(2)}) = N(x) - \frac{\nu_{xy}}{2}h\left[q(x) + \frac{h}{6}\frac{\mathrm{d}p^*}{\mathrm{d}x}\right] + \frac{\nu_{xy}h^2}{6}\frac{\mathrm{d}}{\mathrm{d}x}S_{xy}^{(3)} \tag{16}$$

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}S_{xy}^{(3)} + \frac{12}{h^2}\left[2\nu_{yx} - \frac{E_y}{G_{xy}}\right]\frac{\mathrm{d}^2}{\mathrm{d}x^2}S_{xy}^{(3)} + \frac{504}{h^4}\frac{E_y}{E_x}S_{xy}^{(3)}$$
$$= \frac{21}{10h}\frac{\mathrm{d}^3 q}{\mathrm{d}x^3} + \frac{9}{20}\frac{\mathrm{d}^4 p^*}{\mathrm{d}x^4} + \frac{42}{10h^2}\left(2\nu_{yx} - \frac{E_y}{G_{xy}}\right)\frac{\mathrm{d}^2 p^*}{\mathrm{d}x^2} \tag{17}$$

$$U_x^{(2)} - U_x^{(4)} = \frac{p^* h}{12}\left(\frac{1}{G_{xy}} - \frac{\nu_{yx}}{E_y}\right) - \frac{h^2}{24E_y}\frac{\mathrm{d}q}{\mathrm{d}x} - \frac{h^3}{112E_y}\frac{\mathrm{d}^2 p^*}{\mathrm{d}x^2} + \frac{5h}{21}\left(\frac{\nu_{yx}}{E_y} - \frac{1}{G_{xy}}\right) + \frac{5}{252}\frac{h^3}{E_y}\frac{\mathrm{d}^2}{\mathrm{d}x^2}S_{xy}^{(3)} \tag{18}$$

$$S_x^{(2)} = E_x \frac{\mathrm{d}}{\mathrm{d}x}(U_x^{(2)} - U_x^{(4)}) - \frac{\nu_{xy}h}{12}\frac{\mathrm{d}p^*}{\mathrm{d}x} + \frac{5}{21}h\nu_{xy}\frac{\mathrm{d}}{\mathrm{d}x}S_{xy}^{(3)}$$

$$S_y^{(2)} = -\frac{h}{6}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p^*}{2} - S_{xy}^{(3)}\right) \tag{19}$$

$$S_y^{(4)} = -\frac{h}{14}\frac{\mathrm{d}}{\mathrm{d}x}(S_{xy}^{(3)})$$

$$U_y^{(1)} - U_y^{(3)} = \frac{qh}{4E_y} + \frac{\nu_{yx}}{2E_y}\left[\frac{S_x^{(2)}}{5} - \frac{N}{h}\right] + \frac{h^2}{4E_y}\left[\frac{1}{5}\frac{\mathrm{d}p^*}{\mathrm{d}x} - \frac{3}{7}\frac{\mathrm{d}S_{xy}^{(3)}}{\mathrm{d}x}\right] \tag{20}$$

$$U_y^{(3)} - U_y^{(5)} \approx -\frac{h^2}{120E_y}\frac{\mathrm{d}p^*}{\mathrm{d}x} - \frac{h}{10E_y}\nu_{yx}S_x^{(2)} + \frac{h^2}{36E_y}\frac{\mathrm{d}}{\mathrm{d}x}S_{xy}^{(3)}.$$

At

$x = \bar{x}$, we have

$U_x^{(0)} - U_x^{(2)} = U_A$ or N specified

$U_x^{(2)} - U_x^{(4)}$ or $S_x^{(2)}$ specified (21a)

and

$U_y^{(1)} - U_y^{(3)}$ or $S_{xy}^{(1)} - S_{xy}^{(3)}$ specified.

Bending

$$\frac{\mathrm{d}Q}{\mathrm{d}x}+q^*(x) = 0$$

$$\frac{\mathrm{d}M}{\mathrm{d}x}-Q+\frac{h}{2}p(x) = 0$$

$$\frac{E_x h^3}{12}\frac{\mathrm{d}\beta}{\mathrm{d}x} = M-\frac{h^2}{6}\nu_{xy}\left(\frac{3}{5}q^*+\frac{h}{20}\frac{\mathrm{d}p}{\mathrm{d}x}\right)+\frac{h^3}{60}\nu_{xy}\frac{\mathrm{d}}{\mathrm{d}x}S_{xy}^{(4)} \tag{22}$$

$$\beta+\frac{\mathrm{d}}{\mathrm{d}x}(U_y^{(0)}-U_y^{(2)}) = \frac{6Q}{5G_{xy}h}-\frac{p}{10G_{xy}}+\frac{S_{xy}^{(4)}}{5G_{xy}}+\frac{1}{5}\frac{\mathrm{d}}{\mathrm{d}x}(U_y^{(2)}-U_y^{(4)})$$

$$\begin{aligned}\frac{\mathrm{d}^4}{\mathrm{d}x^4}S_{xy}^{(4)}&+\frac{504}{11h^2}\left(2\nu_{yx}-\frac{E_y}{G_{xy}}\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}S_{xy}^{(4)}+\frac{45360}{11h^4}\frac{E_y}{E_x}S_{xy}^{(4)}\\ &=\frac{3240}{11h^3}\left\{\frac{h}{10}\left(2\nu_{yx}-\frac{E_y}{G_{xy}}\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{p}{2}-\frac{Q}{h}\right]-\frac{h^3}{240}\frac{\mathrm{d}^4p}{\mathrm{d}x^4}+\frac{h^3}{90}\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left(\frac{p}{2}-\frac{Q}{h}\right)\right\}\end{aligned} \tag{23}$$

$$\begin{aligned}U_x^{(3)}-U_x^{(5)} &= \frac{Q}{10}\left(\frac{\nu_{yx}}{E_y}-\frac{1}{G_{xy}}\right)+\frac{ph}{20}\left(\frac{1}{G_{xy}}-\frac{\nu_{yx}}{E_y}\right)+\frac{h^3(1-\nu_{xy}\nu_{yx})}{240E_y}\frac{\mathrm{d}^2p}{\mathrm{d}x^2}\\ &\quad-\frac{h^3}{90E_y}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{p}{2}-\frac{Q}{h}\right)+\frac{7}{45}h\left(\frac{\nu_{yx}}{E_y}-\frac{1}{G_{xy}}\right)S_{xy}^{(4)}+\frac{11}{3240}\frac{h^3}{E_y}\frac{\mathrm{d}^2}{\mathrm{d}x^2}S_{xy}^{(4)}\end{aligned} \tag{24}$$

$$\begin{aligned}S_x^{(3)} &= E_x\frac{\mathrm{d}}{\mathrm{d}x}(U_x^{(3)}-U_x^{(5)})-\frac{\nu_{xy}}{10}h\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{2}-\frac{Q}{h}\right)+\frac{7}{45}\nu_{xy}h\frac{\mathrm{d}}{\mathrm{d}x}S_{xy}^{(4)}\\ S_y^{(3)} &= -\frac{h}{10}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{p}{2}-\frac{Q}{h}-S_{xy}^{(4)}\right]\end{aligned} \tag{25}$$

$$U_y^{(2)}-U_y^{(4)} = \frac{g^*h}{12E_y}+\frac{\nu_{yx}h}{2E_y}\left[\frac{S_x^{(3)}}{7}-\frac{2M}{h^3}\right]+\frac{h^2}{4E_y}\left\{\frac{2}{21}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{2}-\frac{Q}{h}\right)-\frac{1}{9}\frac{\mathrm{d}}{\mathrm{d}x}S_{xy}^{(4)}\right\}. \tag{26}$$

At $x = \bar{x}$

$$\begin{aligned}&U_y^{(0)}-U_y^{(2)} \quad \text{or} \quad Q/h \quad \text{specified}\\ &U_y^{(2)}-U_y^{(4)} \quad \text{or} \quad \left(\frac{p}{2}-\frac{Q}{h}-S_{xy}^{(4)}\right) \quad \text{specified}\end{aligned} \tag{27}$$

and

$$\begin{aligned}&\frac{h\beta}{2} \quad \text{or} \quad \frac{6M}{h^2} \quad \text{specified}\\ &U_x^{(3)}-U_x^{(5)} \quad \text{or} \quad S_x^{(3)} \quad \text{specified.}\end{aligned} \tag{28}$$

Note that we are able to combine and manipulate the equations so that all unknowns are in terms of shear stress functions $S_{xy}^{(3)}$ (for extension) and $S_{xy}^{(4)}$ (for bending); once the shear stress function is obtained, then all unknowns can be determined completely. Note that classical beam theory including transverse normal stress and shear deformation are seen to be just equations (16) and (22) with $S_{xy}^{(3)}$, $S_{xy}^{(4)}$ and $(U_y^{(2)}-U_y^{(4)})$ omitted; the third order theory contains the first correction terms in extension and bending.

The effects of orthotropy, as far as changing stress and deformation distributions, are not present in classical theory. The governing equations for classical beam theory are modified only by the value of the material constants; the *form* of the equations are the same for isotropic and for orthotropic beams. In a higher order theory the effect of orthotropy can severely modify the form of solution as will be seen in the following examples.

APPLICATION OF THEORY

We consider the determination of the elastic distribution for simply supported beams under uniform load, as shown in Fig. 1. This problem has been previously studied in Ref. [4]. We note that $p^*(x) = p(x) = 0$, $q(x) = q^*(x) = -p$; the use of simple support conditions at $x = l/2$ yields

$$Q(x) = px; \qquad M(x) = \frac{p}{2}\left(x^2 - \frac{l^2}{4}\right); \qquad N = 0 \tag{29}$$

and governing equations for the higher order shear stress functions

$$\begin{aligned} &\frac{d^4 S_{xy}^{(3)}}{dx^4} + \frac{12}{h^2}\left[2\nu_{yx} - \frac{E_y}{G_{xy}}\right]\frac{d^2}{dx^2}S_{xy}^{(3)} + \frac{504}{h^4}\frac{E_y}{E_x}S_{xy}^{(3)} = 0 \\ &\frac{d^4 S_{xy}^{(4)}}{dx^4} + \frac{504}{11h^2}\left[2\nu_{yx} - \frac{E_y}{G_{xy}}\right]\frac{d^2}{dx^2}S_{xy}^{(4)} + \frac{45360}{11h^2}\frac{E_y}{E_x}S_{xy}^{(4)} = 0 \end{aligned} \tag{30}$$

Although it is possible that for certain values of the material properties, the characteristic equations will yield repeated roots, in the work to follow we restrict consideration to values for material properties which yield solutions to equation (30) as:

$$S_{xy}^{(3)} = e^{-\alpha_1 x}(c_1 \cos\beta_1 x + c_2 \sin\beta_1 x) + e^{\alpha_1(x-l/2)}[c_3 \cos\beta_1(x-l/2) + c_4 \sin\beta_1(x-l/2)]$$

or

$$S_{xy}^{(3)} = c_1\, e^{-\gamma_1 x} + c_2\, e^{-\gamma_2 x} + c_3\, e^{\gamma_1(x-l/2)} + c_4\, e^{\gamma_2(x-l/2)}$$

$$S_{xy}^{(4)} = e^{-\alpha_2 x}(D_1 \cos\beta_2 x + D_2 \sin\beta_2 x) + e^{\alpha_2(x-l/2)}[D_3 \cos\beta_2(x-l/2) + D_4 \sin\beta_2(x-l/2)]$$

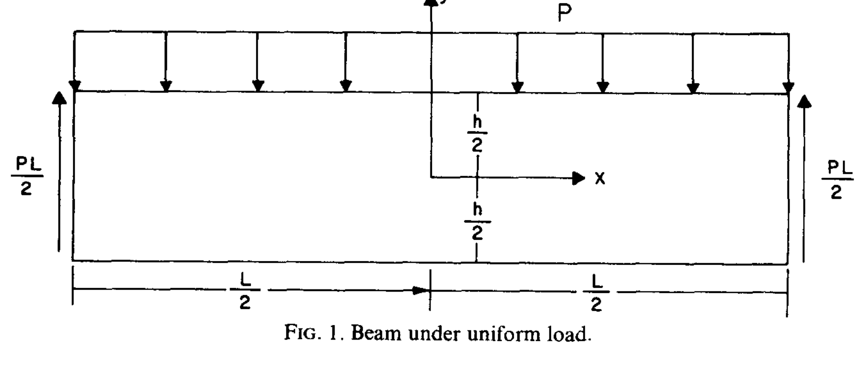

FIG. 1. Beam under uniform load.

or

$$S_{xy}^{(4)} = D_1\,e^{\delta_1 x} + D_2\,e^{-\delta_2 x} + D_3\,e^{\delta_1(x-l/2)} + D_4\,e^{\delta_2(x-l/2)}$$

where α_1, α_2, β_1, β_2, γ_1, γ_2, δ_1, δ_2 are real, positive numbers. For large enough values of $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \delta_1, \delta_2$, we may neglect end interactions when evaluating boundary conditions; to study the form of solutions away from the supports, we need only evaluate C_1, C_2, D_1, D_2. Since $Q = 0$ at $x = 0$, equations (21a) and (27) furnish boundary conditions for the higher order coefficient functions at $x = 0$.

$$S_{xy}^{(3)}(0) = S_{xy}^{(4)}(0) = U_x^{(2)}(0) - U_x^{(4)}(0) = U_x^{(3)}(0) - U_x^{(5)}(0) = 0. \tag{31}$$

Using equations (18) and (24), we may show that the displacement conditions of equation (31) also imply that

$$\frac{d^2}{dx^2}S_{xy}^{(3)}\bigg|_0 = \frac{d^2}{dx^2}S_{xy}^{(4)}\bigg|_0 = 0 \tag{32}$$

so that for this problem, we find

$$S_{xy}^{(3)} = S_{xy}^{(4)} = 0. \tag{33}$$

Therefore, the shear stress solution away from the immediate vicinity of the end support is

$$\sigma_{xy} = -\frac{Q}{h}[P_2(\eta) - 1] = -\frac{3px}{2h}(\eta^2 - 1). \tag{34}$$

Equations (19) and (25) yield

$$S_y^{(2)} = 0; \qquad S_y^{(3)} = \frac{h}{10}\frac{dQ}{dx} = \frac{p}{10}$$

so that the solution for σ_y becomes

$$\sigma_y = \frac{p}{4}[\eta^3 - 3\eta - 2]. \tag{35}$$

With the above solutions, we can also show that

$$S_x^{(2)} = 0; \qquad S_x^{(3)} = \frac{p}{10}\left\{2\nu_{xy} - \frac{E_x}{G_{xy}}\right\} = \frac{\beta ph}{5}$$

so that the solution for $\sigma_x(x, \eta)$ has the form,

$$\sigma_x(x,\eta) = -\frac{3p}{h^2}\left[\frac{l^2}{4} - x^2\right]\eta + \frac{p\beta}{2}\left(\eta^3 - \frac{3}{5}\eta\right). \tag{36}$$

Equations (34)–(36) agree exactly with the solutions of Ref. [4] for orthotropic beams. In Ref. [4] it is shown that the material orthotropy does not cause a *significant* deviation from numerical results obtained using classical theory. Of course, near the supports, the theory developed in this work will yield certain localized additional distributions depending on the exact nature of the support conditions; these effects were not considered in Ref. [4].

As a second and more interesting example, we consider a very localized loading as shown in Fig. 2. This problem has been previously studied in Ref. [6] for the isotropic beam.

We now investigate the stress distribution in the immediate vicinity of the highly localized pressure distribution. In order to obtain numerical data, we use material property values from Ref. [4].

$$E_x = 24{\cdot}3\times 10^6 \text{ psi}; \qquad E_y = 1{\cdot}16\times 10^6 \text{ psi}$$

$$\nu_{xy} = 0{\cdot}252; \qquad G_{xy} = 0{\cdot}44\times 10^6 \text{ psi}$$

$$\nu_{yx} = \frac{E_y\nu_{xy}}{E_x} = 0{\cdot}012.$$

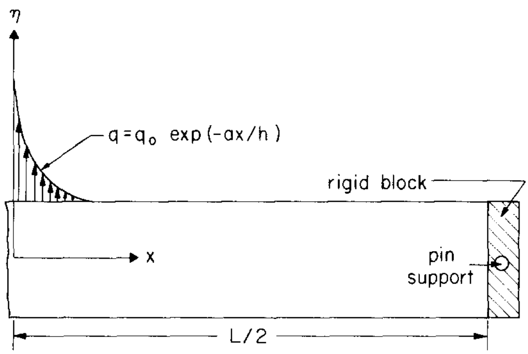

FIG. 2. Beam loading for sample problem.

For the loading considered, governing equations for the higher order shear stress functions are

$$\frac{d^4}{dx^4}S_{xy}^{(3)} - \frac{31{\cdot}348}{h^2}\frac{d^2}{dx^2}S_{xy}^{(3)} + \frac{24{\cdot}059}{h^4}S_{xy}^{(3)} = -\frac{21}{10}\frac{q_0}{h}\left(\frac{a}{h}\right)^3 e^{(-ax/h)} \tag{37}$$

$$\frac{d^4}{dx^4}S_{xy}^{(4)} - \frac{119{\cdot}694}{h^2}\frac{d^2}{dx^2}S_{xy}^{(4)} + \frac{196{\cdot}846}{h^2}S_{xy}^{(4)} = \frac{76{\cdot}95}{h^4}aq_0\, e^{(-ax/h)} - \frac{3{\cdot}27a^3q_0}{h^4}e^{(-ax/h)} \tag{38}$$

which admit the solutions

$$S_{xy}^{(3)} = c_0\, e^{-\gamma_1 x} + c_1\, e^{\gamma_1(x-l/2)} + c_2\, e^{-\gamma_2 x} + c_3\, e^{\gamma_2(x-l/2)} - \frac{21}{10}\frac{q_0}{\mu_1 a}e^{-ax/h}$$

where

$$\gamma_1 = +\frac{0{\cdot}89}{h}$$

$$\gamma_2 = \frac{+5{\cdot}53}{h}$$

$$\mu_1 = 1 - \frac{31{\cdot}348}{a^2} + \frac{24{\cdot}059}{a^4}$$

$$S_{xy}^{(4)} = D_0\, e^{-\delta_1 x} + D_1\, e^{\delta_1(x-l/2)} + D_2\, e^{-\delta_2 x} + D_3\, e^{\delta_2(x-l/2)} + \left(\frac{76{\cdot}95}{a^3} - \frac{3{\cdot}27}{a}\right)\frac{q_0\, e^{(-ax/h)}}{\mu_2}$$

where

$$\delta_1 = \frac{+10{\cdot}86}{h}$$

$$\delta_3 = \frac{+1{\cdot}29}{h}$$

$$\mu_2 = 1 - \frac{119{\cdot}694}{a^2} + \frac{196{\cdot}846}{a^4}.$$

We again neglect end interaction in the evaluation of the integration constants; hence, only C_0, C_2, D_0, D_2 need be evaluated to study the solution near $x = 0$. As in the previous problem, equations (31) provide four boundary conditions to evaluate these constants. Having evaluated the constants, the sheer stress distribution is fully determined near $x = 0$; the remaining stress components are easily determined using the governing equations of the third order theory. For loading parameter "a" = 20, we obtain (near $x = 0$)

$$S_{xy}^{(3)} = [-0{\cdot}0544\,\mathrm{e}^{-0{\cdot}89(x/h)} + 1{\cdot}1957\,\mathrm{e}^{-5{\cdot}53(x/h)} - 1{\cdot}1413\,\mathrm{e}^{-20(x/h)}]\frac{P}{h}$$

$$S_{xy}^{(4)} = [1{\cdot}902\,\mathrm{e}^{-10{\cdot}86(x/h)} + 0{\cdot}296\,\mathrm{e}^{-1{\cdot}29(x/h)} - 2{\cdot}198\,\mathrm{e}^{-20(x/h)}]\frac{P}{h} \quad (39)$$

$$S_{xy}^{(2)} = -\frac{Q}{h}; \qquad Q = \frac{P}{2}[\mathrm{e}^{-20(x/h)} - 1] \quad (40)$$

where

$$P = \frac{2q_0 h}{a}$$

Using equation (28), the shear stress is given as (near $x = 0$)

$$\sigma_{xy} = -\frac{3Q}{2h}(\eta^2 - 1) + \frac{5}{2}S_{xy}^{(3)}\eta(\eta^2 - 1) + \frac{7}{8}S_{xy}^{(4)}(5\eta^4 - 6\eta^2 + 1). \quad (41)$$

Figure 3 shows the shear stress distribution as a function of η at various stations in the vicinity of the loading. Considerable deviation from the parabolic distribution of classical beam theory is observed, although the deviation is essentially established near $x/h = 1{\cdot}0$; at $x/h = 2{\cdot}0$, the shear stress distribution is identical with the predictions of classical beam theory. The higher order solution for isotropic materials, presented in Ref. [6], is essentially equal to the classical beam theory result at $x/h = 0{\cdot}4$; thus, the material orthotropy causes a slower decay of the local higher order shear stress. It is of interest to note that the shear stress behavior is independent of thickness-to-length ratio, a fact which was reported earlier in Ref. [6] for the isotropic beam.

The axial stress distribution has the form

$$\sigma_x = \frac{6M}{h^2}\eta + \frac{S_x^{(2)}}{2}(3\eta^2 - 1) + \frac{S_x^{(3)}}{2}(5\eta^3 - 3\eta) \quad (42)$$

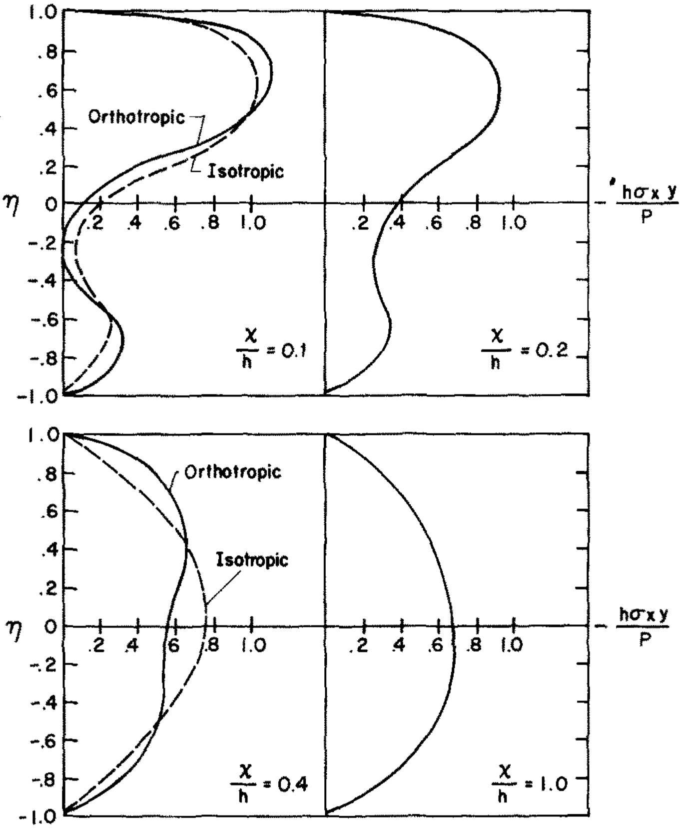

FIG. 3. Shear stress profile near local loading.

where

$$M = \frac{Pl}{4}\left[1-\frac{2x}{l}-\frac{2h}{al}e^{-ax/h}\right]$$

and $S_x^{(2)}$, $S_x^{(3)}$ may be determined from the equations

$$S_x^{(2)} = -\frac{h^2}{12}\frac{E_x}{E_y}\frac{d^2S_y^{(0)}}{dx^2}+\frac{5h}{21}\left[2\nu_{xy}-\frac{E_x}{G_{xy}}\right]\frac{dS_{xy}^{(3)}}{dx}+\frac{70}{3528}\frac{E_x}{E_y}\frac{h^3d^3S_{xy}^{(3)}}{dx^3} \tag{43}$$

$$S_x^{(3)} = \frac{h}{10}\left[\frac{E_x}{E_y}-2\nu_{xy}\right]\frac{dS_{xy}^{(2)}}{dx}-\frac{h^3}{90}\frac{E_x}{E_y}\frac{d^3S_{xy}^{(2)}}{dx^3}$$
$$+\frac{7h}{45}\left(2\nu_{xy}-\frac{E_x}{G_{xy}}\right)\frac{dS_{xy}^{(4)}}{dx}+\frac{E_x}{E_y}\frac{55}{16{,}200}\frac{d^3S_{xy}^{(4)}}{dx^3}. \tag{44}$$

The solution for the axial stress is a function of h/l; also all of the material properties appear explicitly in the solution; in the isotropic solution presented in Ref. [6] the material properties do not appear explicitly. Numerical computations are carried out for h/l = 0·083 (a ratio which may be considered as representing a thin beam); the results are plotted in

Fig. 4 and show that the local loading combined with material orthotropy cause significant deviation from the linear distributions of classical beam theory.

Following Ref. [6], we define an error parameter E as

$$E = \frac{\sigma_{x2} - \sigma_{x0}}{\sigma_{x2}} \times 100$$

where

σ_{x2} = max. axial stress predicted by third order theory

σ_{x0} = max. axial stress predicted by classical beam theory

Table 1 indicates that at $x = 0$ there is a 43 per cent error in the prediction of maximum stress. Note that at $x/h = 1$, the axial stress distribution is nearly linear. The corresponding result for the isotropic beam worked out in Ref. [6] predicted only a maximum error of about 15 per cent.

TABLE 1

x/h	0·0	0·1	1
E	43	22·6	6·9

Thus, our results indicate that in the vicinity of local loadings, material orthotropy has a significant effect on the character of the axial stress. Even for thin beams, where classical theory might seem to be valid, the axial stress distribution is severely modified. For a larger h/l ratio, the effects are even stronger. For $h/l = 0{\cdot}2$, numerical calculations indicate that the maximum axial stress predicted by our third order theory is nearly *three times* the maximum value predicted by a classical theory; in this case, it would seem that even our third order theory is not adequate and additional terms should be computed. The results indicate that for moderately thick orthotropic beams under local loading, the higher order effects may be of a non-negligible character over the entire length of the beam.

The resulting theories are similar to those developed in a previous work for isotropic materials. The slight differences that appear between the theory developed here (when specialized to material isotropy) and the theory developed in Ref. [6] are solely due to interchanging the order of truncation and manipulation of the equations; for practical computations with isotropic materials, the slight changes are of no significance.

These new approximate solutions show that material anisotropy can have a significant effect on the stress distribution in the vicinity of local loadings to the extent that use of any classical beam theory severely underestimates the maximum stress values. However, for slowly varying loadings, the results presented here verify the conclusions drawn in Ref. [4] that material orthotropy can be accounted for within the realm of a classical beam theory.

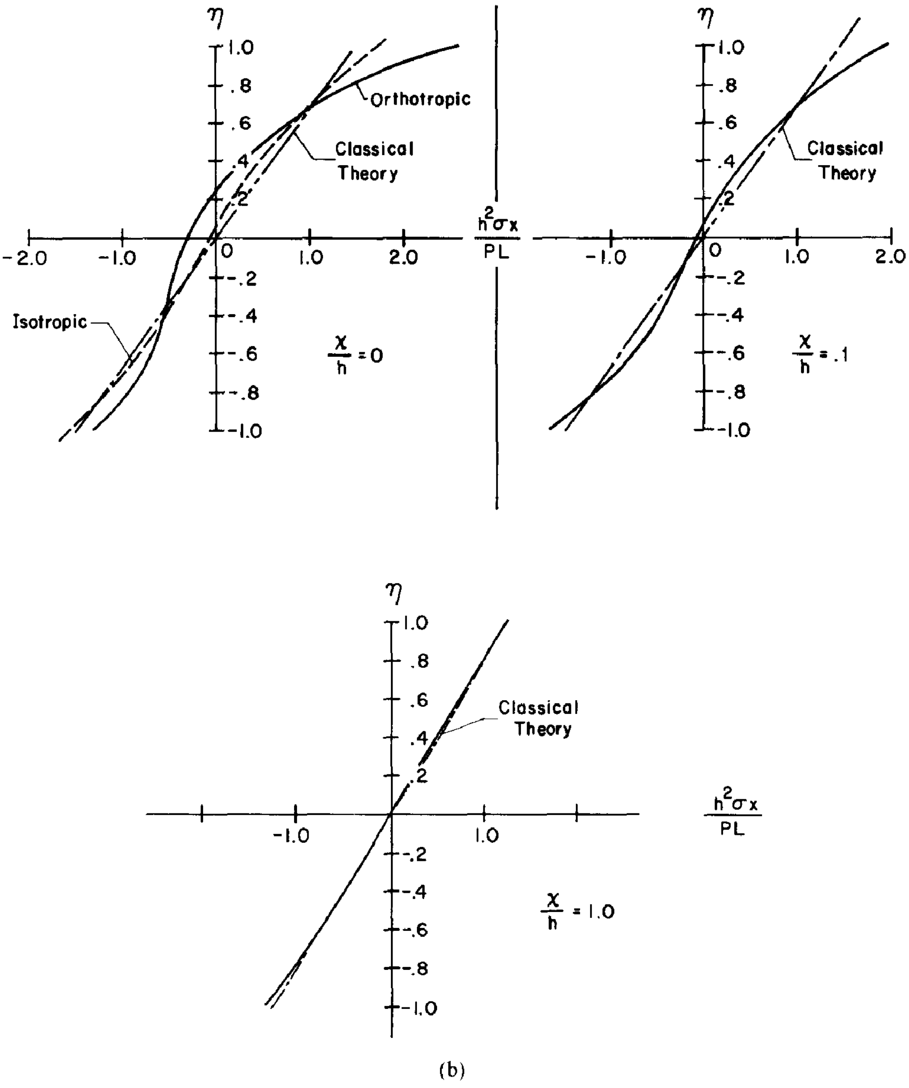

FIG. 4. Axial stress profile $h/L = 0{\cdot}083$.

Finally, we point out that the method of approach developed here can be used to derive higher order structural theories foı plates and shells. It is highly likely that the generalization of our work here to plate and shell structures will show that whenever the effective transverse moduli of a built-up plate or shell structure are significantly less than the in-plane moduli, classical theories may be inadequate to provide an accurate description of behavior under local loadings. We note that the structural behavior of such configurations near "hard points" or boundaries are of interest for various aero- and hydro-space applications; thus, the further development of approximate higher order theories is merited.

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Абстракт—Исследуется определяющие уравнения для ортотропной, прямоугольной полосы с целью получения решений, иллюстрирующих зависимость между классической теорией балки и теориями высших порядков. Точные уравнения обобщенной плоской теории упругости сводятся к сопряженным системам обыкновенных дифференциальных уравнений, путем представления их с помощью рядов в полиномах лежандра, для всех зависимых переменных. Даются сопряженные дифференциальные уравнения в таком виде, чтобы иметь возможность легко определить правильную схему отбрасывания членов, с целью выбора произвольной теории высшего порядка. Разработываются примерные задачи для равномерно и локально загруженных балок, используя при этом типичные значения постоянных материала. Сравнение результатов с существующими решениями указывает на то, что задачи касающиеся медленного изменения нагрузок с ортотропией материала можно применять в смысле классической теории тонкостенных балок. Однако, локальные нагрузки, связанные с ортотропией материала, не дают возможности использовать какой то нибудь классической теории балок, даже для тонкостенных балок.